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## Seven Dimensional Octonionic Yang-Mills Instanton and its Extension to an Heterotic String Soliton

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### Abstract

We construct an octonionic instanton solution to the seven dimensional Yang-Mills theory based on the exceptional gauge group  $G_2$  which is the automorphism group of the division algebra of octonions. This octonionic instanton has an extension to a solitonic solution of the low energy effective theory of the heterotic string that preserves two of the sixteen supersymmetries and hence corresponds to  $N = 1$  space-time supersymmetry in  $(2+1)$  dimensions transverse to the seven dimensions where the Yang-Mills instanton is defined.

Recently there has been a resurgence of interest in solitonic solutions of various string theories, in particular, of the heterotic string [1, 2]. It is hoped that solitonic solutions will shed some light on the non-perturbative dynamics of string theories. Furthermore, they are essential in verifying the validity of generalized duality conjectures in string theories [3]. In this letter our aim is to show that there exists a solution to the self-duality equations in seven dimensional Yang-Mills theory with the gauge group  $G_2$ . The exceptional group  $G_2$  is the automorphism group of the division algebra of octonions  $\mathbf{O}$  [4] and our solution depends in an intricate manner on the existence of this algebra. Hence the term “octonionic instanton”. Eight dimensional octonionic instantons based on the gauge group  $Spin(7)$  were constructed in [5, 6]. To our knowledge the solution we give below is the only known Yang-Mills instanton solution in odd dimensions. It is not difficult to convince oneself that seven is the unique odd dimension where such a solution can exist. In the second part of this paper we extend the seven dimensional Yang-Mills instanton to a solitonic solution of the low energy effective theory of the heterotic string (to order  $\alpha'$ ) in parallel with the work of [7] where the eight-dimensional octonionic instanton was extended to a soliton solution of the heterotic string.

The exceptional group  $G_2$  can be characterized as the maximal common subgroup of the seven dimensional rotation group  $SO(7)$  and its covering group  $Spin(7)$  [4].

$$G_2 = SO(7) \cap Spin(7) \quad (1)$$

For the purposes of this paper we shall regard  $G_2$  as a subgroup of  $Spin(7)$  taken in its eight-dimensional spinor representation which decomposes as

$$\mathbf{8} = \mathbf{7} + \mathbf{1} \quad (2)$$

under its  $G_2$  subgroup. The  $Spin(7)$  generators are given by

$$\Gamma^{mn} = \Gamma^{[m} \Gamma^{n]} \quad (3)$$

where the  $\gamma$  matrices in seven dimensions are defined as usual by

$$\{\Gamma^m, \Gamma^n\} = 2\delta^{mn} \quad (4)$$

with  $m, n, \dots = 1, 2, \dots, 7$ .<sup>1</sup> The generators  $G^{mn} = -G^{nm}$  of  $G_2$  (considered

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<sup>1</sup> Throughout this paper the antisymmetrization  $[m, n, \dots]$  of indices will be of weight one.

as a subgroup of  $Spin(7)$  are defined through the constraints

$$C_{mnp}G^{np} = 0 \quad (5)$$

where  $C_{mnp}$  are the structure constants of the octonion algebra [4, 8]. Since the number of independent constraints is seven, there are altogether 14 generators of  $G_2$ . In terms of the matrices  $\Gamma^{mn}$  the  $G_2$  generators can be written as [8]

$$G^{mn} = \frac{1}{2}\Gamma^{mn} + \frac{1}{8}C_{mnpq}\Gamma^{pq} \quad (6)$$

where the completely antisymmetric tensor  $C^{mnpq}$  is defined as

$$C^{mnpq} := \frac{1}{6}\epsilon^{mnpqrst}C_{rst} \quad (7)$$

These tensors are subject to the identities

$$\begin{aligned} C^{mnp}C_{np}{}^{qr} &= -4C^{mqr} \\ C_{mnp}C^{qrsp} &= 6\delta_{[m}^{[q}C_{n]}^{rs]} \\ C^{mnr}sC_{pqrs} &= 8\delta_{pq}^{mn} - 2C^{mn}{}_{pq} \end{aligned} \quad (8)$$

which can be derived by use of the properties of the structure constants  $C_{mnp}$  and the corresponding identities in eight dimensions [8, 9]. The  $G_2$  commutation relations can be determined from the corresponding commutation relations of  $Spin(7)$  (cf. eq.(11) of [6]) by specializing to its  $G_2$  subgroup. Using the above identities we find after some calculation

$$[G^{mn}, G^{pq}] = 2\delta^{p[n}G^{m]q} - 2\delta^{q[n}G^{m]p} + \frac{1}{2}(C^{pqr[m}G^{n]r} - C^{mnr[p}G^{q]r}) \quad (9)$$

As a check on these relations, readers may verify that the right hand side vanishes upon contraction with either  $C_{smn}$  or  $C_{spq}$ . The constraint (5) implies the identity

$$G_{mn} = \frac{1}{2}C_{mnpq}G^{pq} \quad (10)$$

For later use let us also record the normalization of these generators

$$\text{Tr } G_{mn}G^{pq} = -9P_{mn}{}^{pq} \quad (11)$$

where  $P_{mn}{}^{pq} := \frac{2}{3}(\delta_{mn}^{pq} + \frac{1}{4}C_{mn}{}^{pq})$  is the projector on the  $G_2$  subalgebra of  $Spin(7)$  (with  $\delta_{mn}^{pq} = \delta_{[m}^{[p}\delta_{n]}^{q]}$ ).

Consider now the Yang-Mills gauge theory in seven dimensions with the gauge group  $G_2$ . In analogy with [6] we proceed from the following ansatz for the Yang-Mills gauge field  $A_m(y)$ :

$$A_m(y) = G_{mn}f_n(y) \quad (12)$$

where  $f_n(y) \equiv \partial_n f(y)$  and  $f(y)$  is a scalar function of the coordinates  $y^n$  to be determined by the self-duality condition. The field strength

$$F_{mn} = \partial_m A_n - \partial_n A_m + [A_m, A_n] \quad (13)$$

corresponding to this ansatz takes the form

$$\begin{aligned} F_{mn}(y) = & 2f_{p[m}G_{n]p} - 2f_q G_{q[m}f_{n]} \\ & - G_{mn}(f_q f_q) - \frac{1}{2}C_{mnpq}f_r G_{rq}f_p \end{aligned} \quad (14)$$

where  $f_{pq} \equiv \partial_p \partial_q f(y)$ . We define the dual field strength  $\tilde{F}_{mn}$  using the  $G_2$  invariant tensor  $C_{mnpq}$ :

$$\tilde{F}_{mn}(y) = \lambda C_{mnpq}F_{pq}(y) \quad (15)$$

where  $\lambda$  is a constant to be determined. We get

$$\begin{aligned} \tilde{F}_{mn} = & \lambda \left( 2C_{mnst}f_{ps}G_{tp} - 3C_{mnst}f_q G_{qs}f_t \right. \\ & \left. - 2(f_q f_q)G_{mn} + 4f_q G_{q[m}f_{n]} \right) \end{aligned} \quad (16)$$

To solve the self-duality condition

$$F_{mn} = \tilde{F}_{mn} \quad (17)$$

we put  $f(y) = -\frac{1}{2}\log\phi(y)$ . One finds that  $\lambda = \frac{1}{2}$  and

$$\phi(y) = \rho^2 + r^2 \quad (18)$$

with  $r^2 \equiv y^m y^m$ . Hence the gauge field for this instanton is simply

$$A_m = -\frac{G_{mn}y_n}{\rho^2 + r^2} \quad (19)$$

where  $\rho$  is an arbitrary scale parameter.

Let us now show that the above instanton solution can be extended to a solitonic solution of the heterotic string. Consider the purely bosonic sector of the ten dimensional low energy effective theory of the heterotic string ( to order  $\alpha'$ ).

$$S = \frac{1}{2\kappa^2} \int d^{10}X \sqrt{-g} e^{-2\phi(X)} \left( R + 4(\vec{\nabla}\phi)^2 - \frac{1}{3}H^2 - \frac{1}{30}\alpha' \text{Tr}(F^2) \right) \quad (20)$$

where  $X^M$  ( $M = 0, 1, \dots, 9$ ) denote the coordinates of ten dimensional space-time. We are interested in solutions that preserve at least one supersymmetry. This requires that in ten dimensions there exist at least one Majorana-Weyl spinor  $\epsilon$  such that the supersymmetry variations of the fermionic fields vanish for such solutions [1]

$$\begin{aligned} \delta\chi(X) &= F_{MN}\Gamma^{MN}\epsilon = 0 \\ \delta\lambda(X) &= (\Gamma^M\partial_M\phi - \frac{1}{6}H_{MNP}\Gamma^{MNP})\epsilon = 0 \\ \delta\psi_M(X) &= (\partial_M + \frac{1}{4}\Omega_{-M}^{AB}\Gamma_{AB})\epsilon = 0 \end{aligned} \quad (21)$$

where  $\chi$ ,  $\lambda$  and  $\psi_M$  are the gaugino, dilatino and the gravitino fields, respectively. The generalized connection  $\Omega_-$  is defined as

$$\Omega_{-M}^{AB} = \omega_M^{AB} - H_M^{AB} \quad (22)$$

where  $\omega$  is the spin connection and  $H$  is the anti-symmetric tensor field strength. We denote the ten dimensional world indices as  $M, N, \dots = 0, 1, \dots, 9$  and the corresponding tangent space indices as  $A, B, \dots = 0, 1, \dots, 9$ . Since we are interested in solutions that extend the octonionic instanton we decompose the indices as

$$\begin{aligned} M &= (\alpha, \mu) \quad ; \quad N = (\beta, \nu), \dots \\ A &= (a, m) \quad ; \quad B = (b, n), \dots \\ \alpha, \beta, \dots &= 0, 1, 2 \quad ; \quad \mu, \nu, \dots = 3, 4, \dots, 9 \\ a, b, \dots &= 0, 1, 2 \quad ; \quad m, n, \dots = 3, 4, \dots, 9 \end{aligned} \quad (23)$$

Note that the indices  $m, n, \dots$  that were running from 1 to 7 now run from 3 to 9 so as to agree with the standard convention of denoting the timelike coordinate as  $X^0$ . The coordinates  $y^m$  of the instanton solution will be identified with  $X^m$ . For the purposes of this paper we shall restrict ourselves

to solutions that are Poincare invariant in  $(2+1)$  dimensions. First we choose  $\epsilon$  to be a  $G_2$  singlet of the Majorana-Weyl spinor of  $SO(9,1)$ . There are two such singlets since under the  $G_2 \times SO(2,1)$  subgroup of  $SO(9,1)$  the Majorana-Weyl spinor decomposes as  $\mathbf{16} = (\mathbf{1}, \mathbf{2}) + (\mathbf{7}, \mathbf{2})$ . Let us denote these singlets as  $\eta^i$  ( $i = 1, 2$ ). Thus taking  $\epsilon$  to be a  $G_2$  singlet  $\eta$  and the non-vanishing components of  $F_{MN}$  to be those given by the seven dimensional octonionic instanton the supersymmetry variation  $\delta\chi$  vanishes. This follows from the fact that

$$G^{mn}\eta = 0 \quad (24)$$

and the self-duality of  $F_{mn}$ . The vanishing of the supersymmetry variation  $\delta\lambda$  of the dilatino requires that the non-vanishing components  $H_{mnp}(y)$  of the antisymmetric tensor field strength be related to the dilaton  $\phi$  as follows:

$$H_{mnp} = -\frac{1}{4}C_{mnpq}\partial^q\phi(y) \quad (25)$$

With this choice of  $H_{mnp}$  the gravitino variation  $\delta\psi_M$  also vanishes if we take the metric  $g_{\mu\nu}$  in the seven dimensional subspace to be of the form

$$g_{\mu\nu} = e^{\phi(y)}\delta_{\mu\nu} \quad (\mu, \nu = 3, \dots, 9) \quad (26)$$

with a dilaton field  $\phi(y)$  that is a function of  $y^\mu$ . To solve for this function and thus the dilaton field we need to further impose the Bianchi identity:

$$dH = \alpha' \left( \text{tr} R \wedge R - \frac{1}{30} \text{Tr} F \wedge F \right) \quad (27)$$

where  $\text{Tr}$  refers to the trace in the fundamental representation of  $E_8$  or  $SO(32)$  in the corresponding heterotic string theory. (For  $E_8$  the fundamental representation coincides with the adjoint representation). To order  $\alpha'$  we can neglect the first term [1] and we have

$$dH = -\frac{1}{30}\alpha' \text{Tr} F \wedge F = -\alpha' \text{Tr}_8 F \wedge F \quad (28)$$

where  $\text{Tr}_8$  refers to the trace in the spinor representation of  $Spin(7)$  that contains  $G_2$ . Using

$$\text{Tr} F_{[mn}F_{pq]} = \partial_{[m}\Phi_{npq]} \quad (29)$$

with

$$\Phi_{mnp}(y) := 2 \frac{(3\rho^2 + r^2)}{(\rho^2 + r^2)^3} C_{mnpq} y^q \quad (30)$$

we find

$$e^{-\phi(y)} = e^{-\phi_0} + 4\alpha' \frac{(2\rho^2 + r^2)}{(\rho^2 + r^2)^2} \quad (31)$$

where  $\phi_0$  is the value of the dilaton in the limit  $r \rightarrow \infty$ .

Total ADM mass per unit (d-1)-volume of a (d-1)-brane is given by [10, 2]

$$M_d = \int d^{(10-d)}y \Theta_{00} \quad (32)$$

where  $\Theta_{MN}$  is the total energy-momentum pseudotensor of the combined gravity-matter system. The metric  $g_{mn}(y)$  corresponding to our solution is asymptotically flat with a  $1/r^2$  falloff as  $r \rightarrow \infty$ . This leads to a divergent ADM mass per unit two-brane volume just like the soliton of reference [7] corresponding to the eight dimensional octonionic instanton. The authors of [7] argue that this divergent energy is an infrared phenomenon and does not preclude the existence of a well-behaved low-energy effective action governing the string dynamics on scales large relative to its core size. Their arguments are equally applicable to the soliton solution involving the seven dimensional octonionic instanton we presented above. However, the correct physical interpretation of these solutions and its implications for superstring theory remain to be understood fully.

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## References

- [1] see the review by C.G. Callan, Jr. , J.A. Harvey and A. Strominger, “Supersymmetric String Solitons” in “String Theory and Quantum Gravity ’91” ed. by J. Harvey, R. Iengo, K.S. Narain, S. Randjbar-Daemi and H. Verlinde (World Scientific) and references therein.
- [2] For a very recent review see M.J. Duff, R.R. Khuri and J.X. Liu, “String Solitons” hep-th /9412184.
- [3] For a review see A. Sen, “Strong-Weak Coupling Duality in Four Dimensional String Theory”, TIFR/TH/94-03( hep-th/9402002), and the references therein.
- [4] M. Günaydin and F. Gürsey, *J. Math. Phys.* **14** (1973) 1651-67.
- [5] D.B. Fairlie and J. Nuyts, *J. Phys.* **A17** (1984) 431.
- [6] S. Fubini and H. Nicolai, *Phys.Lett.***155B** (1985) 369.
- [7] J.A. Harvey and A. Strominger, *Phys. Rev. Lett.* **66** (1991) 549.
- [8] B. de Wit and H. Nicolai, *Nucl. Phys.***B231** (1984) 506.
- [9] R. Dündarar, F. Gürsey and C. Tze, *Nucl. Phys.***B266** (1986) 440.
- [10] A. Dabholkar, G. Gibbons, J. A. Harvey and F. Ruiz Ruiz, *Nucl. Phys.***B340** (1990) 33.



## Addendum

### Seven-dimensional octonionic Yang-Mills instanton and its extension to an heterotic string soliton

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Recently we have been informed that self-dual solutions of the Yang-Mills equations for arbitrary gauge groups in dimensions  $d \geq 4$  were already described in

1. A.D. Popov, Europhys. Lett. **17** (1992) 23-26
2. T.A. Ivanova and A.D. Popov, Lett. Math. Phys. **24** (1992) 85-92
3. T.A. Ivanova and A.D. Popov, Theor. Math. Phys. **94** (1993) 225-242
4. T.A. Ivanova, Phys. Lett. **B315** (1993) 277-282

In the last paper the instanton solutions in seven and eight dimensions are extended to heterotic string solitons. While their construction is more general than ours, however, the above authors do not explicitly study the instanton solution of Yang-Mills theory based on the exceptional gauge group  $G_2$ , which was given in our paper and extended to an heterotic string 2-brane.